J. Fluid Mech. (1985), *vol.* 151, *pp.* 279–294
Printed in Great Britain

Slow-drift motion of a two-dimensional block in beam seas

By YEHUDA AGNON
Woods Hole Oceanographic Institution and Massachusetts Institute of Technology

AND CHIANG C. MEI
Massachusetts Institute of Technology

(Received 19 March 1984 and in revised form 29 July 1984)

Because of the inherent nonlinearity in the boundary conditions on the free surface, water waves with frequencies from neighbouring parts of the sea spectrum interact and force low-frequency oscillations at the second order. Since the physical phenomenon involves vastly different timescales, the perturbation method of multiple scales is applied here to a rectangular cylinder in beam seas. It will be shown that at the second order there are two kinds of long waves; one is locked to the envelopes of the incident, reflected and transmitted short waves, while the other propagates away from the body at the long-wave velocity $(gh)^{\frac{1}{2}}$. The latter contributes to the damping of slow-drift oscillations of the body. Analytical results for the displacement amplitudes of the slow sway and for the radiated long waves are derived. The transient evolution due to incident envelopes of finite and semi-infinite duration is also predicted.

1. Introduction

It is well known that, in an irregular sea with a narrow-banded spectrum, second-order effects contribute an exciting force with low frequencies (Hsu & Blenkarn 1970; Remery & Hermans 1971; Newman 1974). As the natural frequency of the moored vessel may also be low, slow-drift oscillation can be excited to cause significant strain in the mooring lines or to affect the dynamic positioning of a floating platform.

Several authors have focused their attention on the long-period exciting force on a fixed body in a narrow-banded stationary sea. Newman has shown that the slowly varying force corresponding to the small frequency difference $\omega_m - \omega_n$, when ω_m and ω_n are two neighbouring frequencies in the spectrum, can be approximately related to the constant drift force for $\omega_m - \omega_n$. Alternatively, Pinkster (1976, 1980) and Faltinsen & Løken (1978) have adopted a straightforward perturbation approach which, in principle, can yield the full second-order solution involving high (sum) and low (difference) frequencies. This is of course necessary if all parts of the second-order theory are desired, or when the sea spectrum is broad. Despite an ingenious use of Green's formula to avoid the full solution, however, the numerical task for practical problems appears to be complex.

Since, among the second-order effects, one is often interested primarily in the slow-drift oscillation, the contrasting frequencies suggest the use of the perturbation method of multiple scales. Steps have been taken in this direction by Molin & Bureau (1980) and Triantafyllou (1982) (see also a recent review by Ogilvie 1983). In their analysis the concept of multiple scales was applied only to time but not to space. It is well known for free waves that slow modulation in time is accompanied by slow modulation in space in the plane of propagation. One should therefore expect the radiation of long waves if the depth is finite. This feature can be most effectively treated by the multiple-scale analysis.

In this paper we apply multiple-scale expansions to both space and time to a two-dimensional body moored in beam seas. After specifying the slowness of modulation, we not only alleviate the need for analysing the short-scale problem, but extend the long-scale problem to include its transient evolution. Thus the incident wave envelope can be finite or semi-infinite in duration. The initial growth, the approach to quasi-steady state, or the final decay are studied through several examples. In addition, two kinds of long waves, one propagating at the group velocity and the other at $(gh)^{\frac{1}{2}}$, are shown to be present. The concept of radiation stress, which is well known in coastal oceanography and engineering, is also brought into the present problem of wave–body interaction.

In order to illustrate the analytical procedure and to examine the physics clearly, we have chosen a very simple geometry and assumed the drift displacement to be small. Large displacement of more practical bodies will be studied along similar lines in the future.

2. Perturbation equations

To demonstrate the analysis, we examine a two-dimensional geometry for which the explicit solution can be simply obtained. Consider a rectangular cylinder sliding without friction on a horizontal sea bottom of depth h (figure 1a). No fluid is permitted to pass above or beneath the cylinder. Potential flow is assumed. The velocity potential $\phi(x, z, t)$ is governed by the Laplace equation

$$\phi_{xx}+\phi_{zz}=0 \quad (-h<z<\zeta), \tag{2.1}$$

with $(\;)_x$ denoting partial differentiation. On the sea bottom the normal flux vanishes:

$$\phi_z=0 \quad (z=-h). \tag{2.2}$$

On the free surface the kinematic condition requires

$$\zeta_t+\zeta_x\phi_x=\phi_z \quad (z=\zeta), \tag{2.3}$$

while the dynamic condition requires

$$g\zeta+\phi_t+\tfrac{1}{2}(\phi_x^2+\phi_z^2)=0 \quad (z=\zeta). \tag{2.4}$$

The rectangular body is assumed to have mass M, and its horizontal movement is partially constrained by an elastic mooring system of spring constant $\bar{K}$. Because there is no communication between the fluids on both sides of the body, the width of the body can be eliminated from the mathematical discussion by using two different coordinate systems: (x_+, z) for $x_+=x-a>0$, and (x_-, z) for $x_-=x+a<0$. More simply, we may drop the subscripts $+$, $-$ and define $x=x_+$ when $x>0$ and $x=x_-$ when $x<0$; the body is reduced mathematically to a thin plate of finite mass (see figure 1b). On the body the exact kinematic and dynamic conditions are respectively

$$X_t=\phi_x \quad (x=X(t)), \tag{2.5}$$

$$\int_{-h}^{\zeta} p(X-0, z, t)\,\mathrm{d}z-\int_{-h}^{\zeta} p(X+0, z, t)\,\mathrm{d}z=MX_{tt}+\bar{K}X \quad (x=X(t)). \tag{2.6}$$

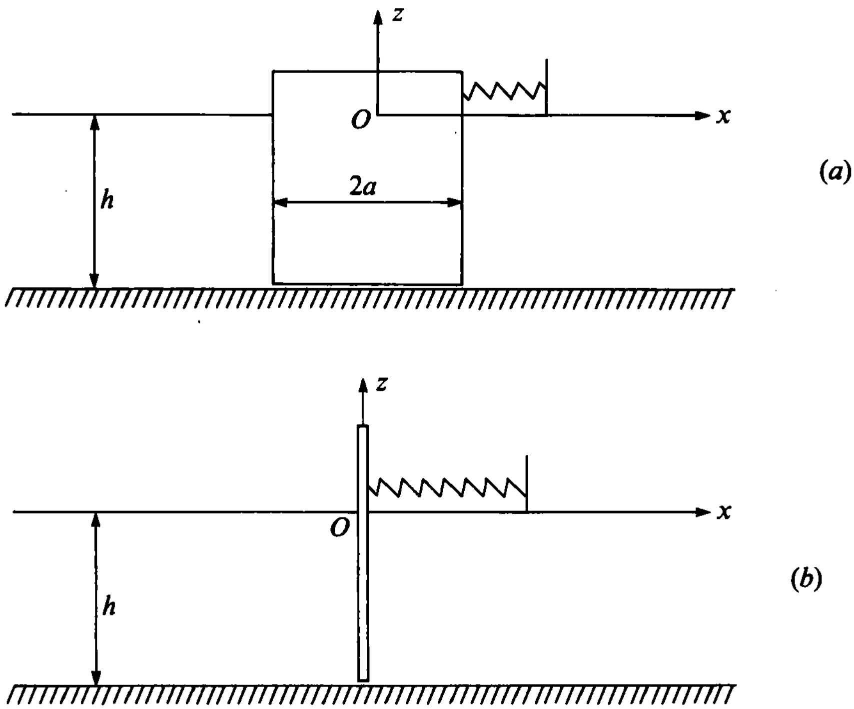

FIGURE 1. Coordinates for (a) a regular block and (b) the equivalent plate.

The pressure is related to the velocity potential, exactly, by the Bernoulli equation:

$$-\frac{p}{\rho} = gz + \phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2). \tag{2.7}$$

In this paper we allow the body displacement X to be no greater than the wave amplitude; the important case where $X \gg O(A)$ will be treated in the future. For small wave amplitudes ($k_0 A = O(\epsilon) \ll 1$, where $k_0 \equiv$ wavenumber) we carry out Taylor expansion about the still-water level at $z = 0$ and combine the two conditions (2.3) and (2.4) to get

$$\phi_{tt} + g\phi_z = \left[-\tfrac{1}{2}(\phi_x^2 + \phi_z^2) + \frac{1}{g}\phi_t \phi_{zt}\right]_t - (\phi_x \phi_t)_x + O(\epsilon^3). \tag{2.8}$$

Similar expansions of (2.5) and (2.6) about $x = 0$ lead to

$$\left.\begin{aligned} X_t &= \phi_x + X\phi_{xx} + O(\epsilon^3), \qquad &(2.9)\\ \Delta \int_{-h}^{\zeta} (p + Xp_x)\,\mathrm{d}z &= MX_{tt} + \tilde{K}X + O(\epsilon^3) \qquad &(2.10) \end{aligned}\right\} \quad (x = 0),$$

where Δ denotes the difference

$$\Delta f \equiv f(x = 0-, t) - f(x = 0+, t). \tag{2.11}$$

Now we assume the incident wavetrain to be nearly periodic at frequency ω and slowly modulated. The length- and timescales of the envelope modulation are $O(\epsilon^{-1})$ times† that of $2\pi/k_0$ and $2\pi/\omega$. Since similar contrast in scales is expected in

† Strictly speaking, the wave slope and the modulation rate are independent, but the present assumption leads to the most interesting situation where nonlinearity and dispersion are competitive.

the response, we introduce expansions in terms of the fast (x, z, t) and the slow $(x_1 = \epsilon x, t_1 = \epsilon t)$ variables:

$$(\phi, \zeta, X) = \epsilon(\phi_1, \zeta_1, X_1) + \epsilon^2(\phi_2, \zeta_2, X_2) + \dots, \tag{2.12}$$

with the convention that

$$\{\phi_n = \phi_n(x, z, t, x_1, t_1),\ \zeta_n = \zeta_n(x, t, x_1, t_1),\ X_n = X_n(t, t_1)\} = O(1). \tag{2.13}$$

To ensure that $X = O(\epsilon)$ we must keep

$$M = O\left(\frac{\rho g h}{\omega^2}\right) = O\left(\frac{\rho h^2}{k_0 h}\right) = O(\rho h^2). \tag{2.14}$$

This restriction is satisfied by vessels of ordinary size. Since the slow-drift exciting force is of the second order in $k_0 A$ and the corresponding displacement is allowed to be of the first order, we must have a weak mooring such that

$$\tilde{K} = O(\epsilon) = \epsilon K, \tag{2.15}$$

where K is of order unity. In practice, the mooring system can vary widely and K can be of any value. Using (2.11) and (2.13), we obtain, from the Laplace equation in $-h < z < 0$

$$\phi_{1xx} + \phi_{1zz} = 0, \tag{2.16a}$$

$$\phi_{2xx} + \phi_{2zz} = -2\phi_{1xx_1}, \tag{2.16b}$$

$$\dots,$$

from the bottom condition on $z = -h$

$$\phi_{1z} = \phi_{2z} = \dots = 0, \tag{2.17}$$

and from the free-surface condition on $z = 0$

$$\phi_{1tt} + g\phi_{1z} = 0, \tag{2.18a}$$

$$\phi_{2tt} + g\phi_{2z} = -2\phi_{1tt_1} - \left[\tfrac{1}{2}(\phi_{1x}^2 + \phi_{1z}^2) + \frac{1}{g}(\phi_{1t}\,\phi_{1z})_t\right]_t - (\phi_{1x}\,\phi_{1t})_x, \tag{2.18b}$$

$$\dots .$$

The kinematic condition on the body $(x = 0)$ leads to

$$X_{1t} = \phi_{1x}, \tag{2.19a}$$

$$X_{1t_1} + X_{2t} = \phi_{1x_1} + \phi_{2x} + X_1\,\phi_{1xx}. \tag{2.19b}$$

$$\dots .$$

The dynamic condition on the body gives

$$-\rho\,\Delta \int_{-h}^{0} \phi_{1t}\,\mathrm{d}z = M X_{1tt}, \tag{2.20a}$$

$$-\rho\,\Delta \int_{-h}^{0} \mathrm{d}z\,[\phi_{2t} + \phi_{1t_1} + X_1\,\phi_{1tx} + \tfrac{1}{2}(\phi_{1x}^2 + \phi_{1z}^2)] - \rho\,\Delta(\zeta_1\,\phi_{1t} + \tfrac{1}{2}g\zeta_1^2)_{z=0} = M(2X_{1tt_1} + X_{2tt}) + K X_1. \tag{2.20b}$$

Use can be made of (2.19a) to eliminate terms on the left of (2.20b).

The solution is sought in terms of harmonics with respect to the fast time, i.e.

$$(\phi_n, \zeta_n, X_n) = \sum_{m=-n}^{n} (\phi_{nm}, \zeta_{nm}, X_{nm})\,\mathrm{e}^{-\mathrm{i}m\omega t}, \tag{2.21}$$

with $$\phi_{nm} = \phi^*_{n,-m}, \quad \text{etc.},$$

where ( )* denotes the complex conjugate. We now examine each order in turn.

3. The short-scale motion

The first-order problem is not affected by the weak spring, so that the horizontal motion is unconstrained. The first-harmonic potential ϕ_{11} obeys the following equations:

$$\phi_{11xx} + \phi_{11zz} = 0 \quad (-h < z < 0), \tag{3.1}$$

$$\phi_{11z} - \sigma\phi_{11} = 0 \quad (z = 0), \quad \text{where } \sigma \equiv \frac{\omega^2}{g}, \tag{3.2}$$

$$\phi_{11z} = 0 \quad (z = -h), \tag{3.3}$$

$$-\mathrm{i}\omega X_{11} = \phi_{11x} \quad (x = 0\pm, \ -h < z < 0), \tag{3.4}$$

$$-\Delta \int_{-h}^{0} \phi_{11}\, \mathrm{d}z = \frac{M}{\rho}\phi_{11x} \quad (x = 0, \ -h < z < 0). \tag{3.5}$$

In getting (3.5), (3.4) is used. In addition, the disturbance induced by the incident waves must be outgoing at infinity. The solution that satisfies (3.1)–(3.3) and the radiation condition can be expressed as follows:

$$\phi_{11} = \begin{cases} (a_0^+ - b_0^+) f_0(z)\, \mathrm{e}^{\mathrm{i}k_0 x} + \sum\limits_{n=1}^{\infty} b_n^+ f_n(z)\, \mathrm{e}^{-k_n x} & (x > 0), \\ (a_0^- \,\mathrm{e}^{\mathrm{i}k_0 x} + b_0^- \,\mathrm{e}^{-\mathrm{i}k_0 x}) f_0(z) + \sum\limits_{n=1}^{\infty} -b_n^- f_n \,\mathrm{e}^{k_n x} & (x < 0), \end{cases} \tag{3.6}$$

where

$$\left.\begin{aligned} f_0 &= \surd 2 \cosh k_0(z+h)\,(h+\sigma^{-1}\sinh^2 k_0 h)^{-\frac{1}{2}}, \\ f_n &= \surd 2 \cos k_n(z+h)\,(h-\sigma^{-1}\sin^2 k_n h)^{-\frac{1}{2}}, \end{aligned}\right\} \tag{3.7}$$

with $$k_0 \tanh k_0 h = \sigma, \quad k_n \tan k_n h = -\sigma. \tag{3.8}$$

Note that $\{f_0, f_1, f_2, \ldots\}$ form an orthonormal set in the range $-h < z < 0$ and that

$$a_0^\pm = a_0^\pm(x_1, t_1), \quad b_0^\pm = b_0^\pm(x_1, t_1), \quad b_n^\pm = b_n^\pm(t_1). \tag{3.9}$$

The terms associated with a_0^-, b_0^- and $a_0^+ - b_0^+$ represent respectively the incident, reflected and transmitted waves. The corresponding free-surface displacement of the incident waves is

$$\zeta_1 = \zeta_{11}\,\mathrm{e}^{-\mathrm{i}\omega t} + * = \tfrac{1}{2}\mathscr{A} \exp \mathrm{i}(k_0 x - \omega t + \delta) + *, \tag{3.10}$$

which defines the first-order wave amplitude and phase lag δ:

$$\mathscr{A}\mathrm{e}^{\mathrm{i}\delta} = \frac{2\mathrm{i}\omega}{g} a_0^- f_0(0). \tag{3.11}$$

By using (3.4), we find

$$a_0^-(0, t_1) = a_0^+(0, t_1) \equiv A_0(t_1), \tag{3.12a}$$

$$b_0^-(0, t_1) = b_0^+(0, t_1) \equiv B_0(t_1), \tag{3.12b}$$

$$b_n^+(t_1) = b_n^-(t_1) \equiv B_n(t_1). \tag{3.12c}$$

When (3.6) is substituted into (3.5) and the scalar product of the resulting equation is taken with f_n for each n, we get

$$A_0 - B_0 = -\frac{2\mathrm{i}\rho}{k_0 M}\left(-B_0 F_0 + \sum_{m=1}^{\infty} B_m F_m\right) F_0 \tag{3.13a}$$

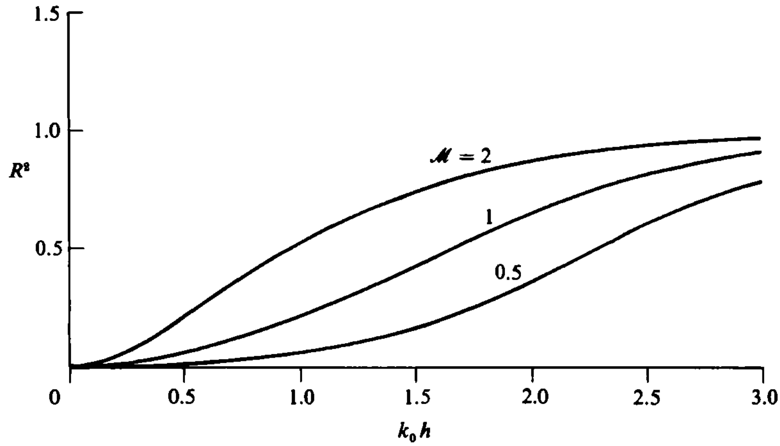

FIGURE 2. Square of reflection coefficient for a moving block for various mass ratios $\mathcal{M} = M/\rho h^2$.

and
$$B_n = -\frac{2\rho}{k_n M} F_n \left(-B_0 F_0 + \sum_{m=1}^{\infty} B_m F_m\right), \tag{3.13b}$$

where
$$F_n \equiv \int_{-h}^{0} f_n \, \mathrm{d}z. \tag{3.14}$$

The simultaneous equations can be solved to give

$$B_0 = A_0 \left\{1 + \frac{\mathrm{i}F_0^2}{k_0} \Big/ \left[\Sigma \frac{F_n^2}{k_n} + \frac{M}{2\rho}\right]\right\}^{-1}, \tag{3.15a}$$

$$B_n = A_0 \frac{F_n F_0}{k_n} \left[\frac{\mathrm{i}F_0^2}{k_0} + \Sigma \frac{F_n^2}{k_n} + \frac{M}{2\rho}\right]^{-1}. \tag{3.15b}$$

The reflection coefficient may be defined as

$$R \equiv |B_0/A_0|, \tag{3.16}$$

which is a pure constant. Typical values of R^2 are plotted in figure 2. Note that B_0 incorporates waves reflected by the presence, and radiated by the induced motion, of the body. Note also that $B_n(t_1)/A_0(t_1)$ are independent of t_1 for all n. The first-order first-harmonic amplitude of the body displacement $\mathscr{X}_{11} \equiv |X_{11}/\mathscr{A}|$ is plotted in figure 3 for a wide range of $k_0 h$ and for three values of $\mathcal{M} \equiv M/\rho h^2$. The weak mooring has no effect on the first-order short-scale motion (hence R).

From (3.12) it may be shown that

$$\mathrm{Re}\,(A_0 - B_0) B_0^* = 0, \tag{3.17a}$$

so that
$$|A_0 - B_0|^2 = |A_0|^2 - |B_0|^2, \tag{3.17b}$$

which implies energy conservation.

To get the long-scale variation of a_0 and b_0 we must examine the second-order first-harmonic problem, whose governing equations are

$$\left.\begin{aligned} \phi_{21xx} + \phi_{21zz} &= 0 \quad (-h < z < 0), \\ \phi_{21z} - \sigma\phi_{21} &= \frac{2\mathrm{i}\omega}{g}\phi_{11t_1} \quad (z = 0), \\ \phi_{21z} &= 0 \quad (z = -h). \end{aligned}\right\} \tag{3.18}$$

In the region $|k_0 x| \gg 1$ and $O(x_1) = 1$ the evanescent modes are insignificant. In order that propagating modes may exist, a solvability condition (see e.g. Mei 1983) must be satisfied. This leads to

$$\left(\frac{\partial}{\partial t_1} \pm C_g \frac{\partial}{\partial x_1}\right)\begin{pmatrix} a_0^- \\ b_0^- \end{pmatrix} = 0, \quad \left(\frac{\partial}{\partial t_1} + C_g \frac{\partial}{\partial x_1}\right)(a_0^+ - b_0^+) = 0, \tag{3.19}$$

which imply

$$a_0^{\pm} = A_0(x_1 - C_g t_1), \quad b_0^{\pm} = B_0(x_1 \mp C_g t_1). \tag{3.20}$$

The first-order solution is complete.

4. The long-scale motion

To predict the long-scale displacement $X_{10}(t_1)$ of the body, we only need ϕ_{10}, ζ_{10} being identically zero. From the zeroth harmonic of (2.16*a*)–(2.19*a*), the short-scale variation of ϕ_{10} is governed by the following equations:

$$\left.\begin{aligned} &\phi_{10xx} + \phi_{10zz} = 0, \\ &\phi_{10z} = 0 \quad (z = 0, -h) \\ &\phi_{10x} = 0 \quad (x = 0\pm). \end{aligned}\right\} \tag{4.1}$$

Hence $\phi_{10} = \phi_{10}(x_1, t_1)$ is independent of short scales. The long-scale equations for ϕ_{10} and the corresponding mean sea level ζ_{20} are most simply obtained from the zeroth harmonic of the continuity equation

$$\zeta_t + \left(\int_{-h}^{\zeta} u \, \mathrm{d}z\right)_x = 0. \tag{4.2}$$

and the Bernoulli equation in the region $x_1 = O(1)$. Defining by U the flux normalized by the still water depth h,

$$U = \frac{1}{h}\int_{-h}^{\zeta} u \, \mathrm{d}z = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} U_{nm} \, \mathrm{e}^{-\mathrm{i}m\omega t}, \tag{4.3}$$

we obtain, after expanding the integral,

$$hU_{20} = h\phi_{10x_1} + (\zeta_{11}^* \phi_{11x} + *)\,|_{z=0}. \tag{4.4}$$

Now the quadratic terms are made up of self-products of rightgoing, leftgoing ($x < 0$ only), and evanescent waves, as well as cross-products of counterpropagating waves ($x < 0$), and of evanescent and propagating waves. The self-products of propagating waves (right- or leftgoing) depend on x_1 and t_1, but not on the short scales, and are responsible for ϕ_{10}. Products containing evanescent modes die out, while cross-products of counterpropagating waves die out upon spatial average, in the region $x_1 = O(1)$. If an overbar is used to distinguish the self-product of propagating waves and their responses from the rest, then we obtain from (4.2), at the zeroth harmonic and third order,

$$\bar{\zeta}_{20t_1} + h\phi_{10x_1x_1} + \overline{(\zeta_{11}^* \phi_{11x} + *)_{x_1}} = 0. \tag{4.5}$$

Similarly, from the Bernoulli equation we have

$$-g\bar{\zeta}_{20} = \phi_{10t_1} + \overline{|\phi_{11x}|^2} + \overline{|\phi_{11z}|^2} - \overline{\sigma(\phi_{11}\phi_{11z}^* + *)} \quad (z = 0). \tag{4.6}$$

Finally, by differentiation we eliminate ζ_{20} from (4.5) and (4.6) to get

$$\begin{aligned} \phi_{10t_1t_1} - gh\phi_{10x_1x_1} &= \overline{g(\zeta_{11}^* \phi_{11x} + *)}_{x_1} \\ &\quad - \{\overline{|\phi_{11x}|^2} + \overline{|\phi_{11z}|^2} - \overline{\sigma(\phi_{11}\phi_{11z}^* + *)}\}_{t_1}\,|_{z=0} \quad (x_1 \gtrless 0). \end{aligned} \tag{4.7}$$

With lengthier algebra, the same governing equation can also be obtained more formally at the third order by requiring the solvability of the boundary-value problem for ϕ_{30}.

From the kinematic and dynamic boundary conditions on the body we have

$$X_{10t_1} = \phi_{10x_1} + \phi_{20x} + (X_{11}^* \phi_{11xx} + *), \tag{4.8}$$

$$-\frac{K}{\rho} X_{10} = \rho\,\Delta \int_{-h}^{0} \mathrm{d}z\,[\phi_{10t_1} + |\phi_{11x}|^2 + |\phi_{11z}|^2] + \rho\,\Delta[(-\mathrm{i}\omega\phi_{11}\zeta_{11}^* + *) + g|\zeta_{11}|^2]|_{z=0}. \tag{4.9}$$

In addition, ϕ_{10} must satisfy the radiation condition.

In view of (3.20), the forcing terms in (4.7) are functions of $x_1 - C_g t_1$ (transmitted envelope) for $x_1 > 0$. We therefore expect ϕ_{10} to be of the form

$$\phi_{10} = \phi_{10}^{\mathrm{T}}(x_1 - C_g t_1) + \phi_{10}^{+}(x_1 - (gh)^{\frac{1}{2}} t_1) \quad (x_1 > 0). \tag{4.10}$$

The second term on the right is a homogeneous solution representing a long wave propagating at the speed $(gh)^{\frac{1}{2}}$. Similarly, for $x_1 < 0$ the forcing terms depend either on $x_1 - C_g t_1$ (incident envelope) or on $x_1 + C_g t_1$ (reflected envelope). We must then have

$$\phi_{10} = \phi_{10}^{\mathrm{I}}(x_1 - C_g t_1) + \phi_{10}^{\mathrm{R}}(x_1 + C_g t_1) + \phi_{10}^{-}(x_1 + (gh)^{\frac{1}{2}} t_1) \quad (x_1 < 0). \tag{4.11}$$

The particular solutions $\phi_{10}^{(\alpha)}$, $(\alpha) = \mathrm{I}$, T or R, can be readily calculated and have the following result:

$$\phi_{10x_1}^{(\alpha)} = \psi^{(\alpha)} \frac{f_0^2(0)}{C_g^2 - gh} [(k_0^2 - \sigma^2) C_g + 2\omega k_0], \tag{4.12a}$$

where

$$\psi^{\mathrm{I}} = |a_0^-|^2, \quad \psi^{\mathrm{T}} = |a_0^+ - b_0^+|^2, \quad \psi^{\mathrm{R}} = -|b_0^-|^2. \tag{4.12b}$$

Imposing the initial condition that there is no disturbance at the body before the incident envelope arrives, we get

$$\phi_{10x_1}^{(\alpha)} = 0 \quad \text{at } t_1 = 0, x_1 = 0. \tag{4.13}$$

The homogeneous solutions $\phi_{10}^{\pm}$ remain to be determined.

For later use we need the long-scale parts $\bar{U}_{20}$ and $\bar{\zeta}_{20}$. With a little algebra, (4.4) gives

$$\begin{aligned} \bar{U}_{20}^{(\alpha)} &= \phi_{10x_1}^{(\alpha)} + \frac{1}{h}[\phi_{11x}^{(\alpha)} \zeta_{11}^{(\alpha)*} + *]_{z=0} \\ &= \phi_{10x_1}^{(\alpha)} + \psi^{(\alpha)} \frac{2\omega k_0}{gh} f_0^2(0) \quad \text{for } (\alpha) = \mathrm{I}, \mathrm{T}, \mathrm{R}. \end{aligned} \tag{4.14}$$

When (4.12) and (4.14) are combined, we obtain

$$\left.\begin{matrix} \bar{U}_{20}^{\mathrm{I}} \\ \bar{U}_{20}^{\mathrm{T}} \\ \bar{U}_{20}^{\mathrm{R}} \end{matrix}\right\} = \frac{C_g}{\rho h} \frac{1}{gh - C_g^2} \begin{cases} -S(x_1 - C_g t_1), & (4.15a) \\ -(1 - R^2) S(x_1 - C_g t_1), & (4.15b) \\ R^2 S(x_1 + C_g t_1), & (4.15c) \end{cases}$$

where

$$S(\xi) = \frac{\rho}{2} g \mathscr{A}^2(\xi) \left(\frac{2C_g}{C} - \frac{1}{2}\right) \tag{4.16}$$

is the x-component of the radiation stress in a progressive wavetrain with the phase function ξ. On the two sides of the body the total drift velocities are

$$\bar{U}_{20} = \bar{U}_{20}^{\mathrm{I}} + \bar{U}_{20}^{\mathrm{R}} + \bar{U}_{20}^{-} \quad (x_1 < 0), \tag{4.17a}$$

$$\bar{U}_{20} = \bar{U}_{20}^{\mathrm{T}} + \bar{U}_{20}^{+} \quad (x > 0). \tag{4.17b}$$

The corresponding mean sea level can be decomposed similarly, with

$$\left.\begin{matrix}\bar{\zeta}^{\mathrm{I}}_{20} \\ \bar{\zeta}^{\mathrm{T}}_{20} \\ \bar{\zeta}^{\mathrm{R}}_{20}\end{matrix}\right\} = \frac{h}{C_{\mathrm{g}}}\left\{\begin{matrix}\bar{U}^{\mathrm{I}}_{20}, \\ \bar{U}^{\mathrm{T}}_{20} \\ -\bar{U}^{\mathrm{R}}_{20},\end{matrix}\right. \quad \text{and} \quad \bar{\zeta}^{\pm}_{20} = \pm\frac{h}{(gh)^{\frac{1}{2}}}\bar{U}^{\pm}_{20}, \qquad \begin{matrix}(4.18a) \\ (4.18b) \\ (4.18c)\end{matrix}$$

which follows from continuity. For unidirectional waves, formulae corresponding to (4.15) are well-known (Longuet-Higgins & Stewart 1964). It is worth pointing out that

$$\bar{\zeta}^{\mathrm{I}}_{20} - \bar{\zeta}^{\mathrm{R}}_{20} = \bar{\zeta}^{\mathrm{T}}_{20} \tag{4.19}$$

for $x_1 = 0$ and all $t_1 > 0$.

To find $\phi^{\pm}_{10}$, and hence $\bar{\zeta}^{\pm}_{20}$ and $\bar{U}^{\pm}_{20}$, we must turn to the boundary conditions on the body.

Integrating the kinematic boundary condition (2.19) from $-h$ to 0 and using

$$\int_{-h}^{0} \phi_{11xx}\,\mathrm{d}z = -\int_{-h}^{0} \phi_{11zz}\,\mathrm{d}z = -\sigma\phi_{11}|_{z=0},$$

we get

$$hX_{10t_1} = h\phi_{10x_1} + \int_{-h}^{0}\frac{\partial\phi_{20}}{\partial x}\,\mathrm{d}z + \left[\frac{\mathrm{i}\omega}{g}(\phi_{11}\phi^*_{11x})_{z=0} + *\right] \quad (x = 0\pm). \tag{4.20}$$

We now insist that ϕ_{20} is bounded as $|x|\to\infty$ ($|x_1| \ll 1$ still), implying that all the long-spatial-scale motion is taken up by ϕ_{10}. Then the integral in (4.20) must balance the short-scale part (evanescent modes) of the quadratic terms, i.e.

$$X_{10t_1} = \phi_{10x_1} + \frac{1}{gh}[\mathrm{i}\omega\phi_{11}\phi^*_{11x} + *]_{|x|\to\infty,\, z=0}. \tag{4.21}$$

This can be more formally deduced as follows. The short-scale dependence of ϕ_{20} is governed by the following conditions:

$$\left.\begin{aligned}&\phi_{20xx} + \phi_{20zz} = 0, \\ &\phi_{20z} = \frac{1}{g}(\mathrm{i}\omega\phi_{11}\phi^*_{11x} + *)_x \quad (z = 0), \\ &\phi_{20z} = 0 \quad (z = -h).\end{aligned}\right\} \tag{4.22}$$

Consider a control volume to the right of the body and bounded by the horizontal lines $z = 0$ and $z = -h$, and by the vertical lines $x = 0$ and $x\to\infty$, the last of which is in the region $O(x_1) \ll 1$ but outside the influence of the evanescent modes, i.e. $k_0|x| \gg 1$. Applying Green's formula to ϕ_{10} (or any constant) and ϕ_{20}, we get

$$\int_{-h}^{0}\phi_{20x}\,\mathrm{d}z\bigg|_{x=0} = \int_{-h}^{0}\phi_{20x}\,\mathrm{d}z\bigg|_{x\to\infty} + \frac{1}{g}\{[(-\mathrm{i}\omega\phi_{11}\phi^*_{11x} + *)]_{z=0}\}^{x=0}_{x\to\infty}. \tag{4.23}$$

To avoid linear growth of ϕ_{20} in x we discard the first term on the right, yielding

$$\int_{-h}^{0}\phi_{20x}\,\mathrm{d}z\bigg|_{x=0} - \frac{1}{g}[-\mathrm{i}\omega\phi_{11}\phi^*_{11x} + *]_{x=0,\, z=0} = -\frac{1}{g}[-\mathrm{i}\omega\phi_{11}\phi^*_{11x} + *]_{x\to\infty,\, z=0}. \tag{4.24}$$

Similar reasoning for a control volume to the left of the body gives (4.24) also, with the right-hand term above being evaluated at $x\to-\infty$. Hence (4.21) follows.

The second term on the right of (4.21) can be computed for both $x\to\pm\infty$ to be

$$\frac{2\omega k_0}{gh}(|A_0|^2 - |B_0|^2),$$

which is the same as
$$= \frac{1}{gh}\,[\overline{\mathrm{i}\omega\phi_{11}\,\phi^*_{11x}+*}]_{|x|\to\infty,\,z=0}$$
$$= \frac{1}{h}\,[\overline{\zeta_{11}\,\phi^*_{11x}+*}]_{|x|\to\infty,\,z=0}.$$

Using (4.4) and (4.24), we can write
$$X_{10t_1} = \begin{cases} \bar{U}^{\mathrm{I}}_{20}+\bar{U}^{\mathrm{R}}_{20}+\bar{U}^{-}_{20} & (x=0-), \\ \bar{U}^{\mathrm{T}}_{20}+\bar{U}^{+}_{20} & (x=0+). \end{cases} \tag{4.25}$$

In view of (4.18), we further have
$$\bar{\zeta}^{-}_{20} = -\frac{h}{(gh)^{\frac{1}{2}}}\,\bar{X}_{10t_1}+\frac{C_{\mathrm{g}}}{(gh)^{\frac{1}{2}}}\,(\bar{\zeta}^{\mathrm{I}}_{20}-\bar{\zeta}^{\mathrm{R}}_{20}) \quad (x_1=0-), \tag{4.26a}$$
$$\bar{\zeta}^{+}_{20} = \frac{h}{(gh)^{\frac{1}{2}}}\,\bar{X}_{10t_1}-\frac{C_{\mathrm{g}}}{(gh)^{\frac{1}{2}}}\,\bar{\zeta}^{\mathrm{T}}_{20} \quad (x_1=0+). \tag{4.26b}$$

Note that
$$\bar{\zeta}^{-}_{20} = -\bar{\zeta}^{+}_{20} \quad (x_1=0) \tag{4.27}$$
because of (4.19).

The dynamic condition (4.9) may be simplified after using the solutions for ϕ_{11} and ζ_{11}, with the result
$$\frac{K}{\rho}X_{10} = -h\,\Delta\phi_{10t_1}+\frac{C_g}{C}R^2 g\mathscr{A}^2. \tag{4.28}$$

Now,
$$\begin{aligned}\Delta\phi_{10t_1} &= (\phi^{\mathrm{I}}_{10}+\phi^{\mathrm{R}}_{10}+\phi^{-}_{10}-\phi^{\mathrm{T}}_{10}-\phi^{+}_{10})_{t_1} \\ &= [-C_{\mathrm{g}}(\phi^{\mathrm{I}}_{10}-\phi^{\mathrm{R}}_{10}-\phi^{\mathrm{T}}_{10})+(gh)^{\frac{1}{2}}(\phi^{-}_{10}+\phi^{+}_{10})]_{x_1} \\ &= -C_{\mathrm{g}}\left\{(U^{\mathrm{I}}_{20}-U^{\mathrm{R}}_{20}-U^{\mathrm{T}}_{20})-\frac{1}{h}\,[(\phi^{\mathrm{I}}_{11x}\,\zeta^{\mathrm{I}*}_{11x}+*)-(\phi^{\mathrm{R}}_{11x}\,\zeta^{\mathrm{R}*}_{11}+*)-(\phi^{\mathrm{T}}_{11x}\,\zeta^{\mathrm{T}*}_{11}+*)]\right\} \\ &\quad +(gh)^{\frac{1}{2}}(\bar{U}^{-}_{20}+\bar{U}^{+}_{20}) \quad (x_1=0),\end{aligned} \tag{4.29}$$
where (4.9), (4.11) and (4.18) have been invoked. The terms involving $\phi^{(\alpha)}_{11}$ and $\zeta^{(\alpha)}_{11}$ cancel with the last term in (4.28), so that
$$\frac{K}{\rho h}X_{10} = C_{\mathrm{g}}(\bar{U}^{\mathrm{I}}_{20}-\bar{U}^{\mathrm{R}}_{20}-\bar{U}^{\mathrm{T}}_{20})-(gh)^{\frac{1}{2}}(\bar{U}^{-}_{20}+\bar{U}^{+}_{20}). \tag{4.30}$$

With the help of (4.25), $\bar{U}^{-}_{20}$ and $\bar{U}^{+}_{20}$ are eliminated, yielding finally
$$X_{10t_1}+\frac{K}{2\rho h(gh)^{\frac{1}{2}}}\,X_{10}+\left[\frac{C_{\mathrm{g}}}{(gh)^{\frac{1}{2}}}\,R^2+(1-R^2)\right]\frac{C_{\mathrm{g}}}{\rho h}\,\frac{S}{gh-C^2_{\mathrm{g}}} = 0, \tag{4.31}$$
where $S = S(-C_{\mathrm{g}}\,t_1)$. This is the differential equation for the slow sway of the body. The first term signifies the effect of radiation damping, the second term the effect of mooring, and the third term, the wave momentum flux. Because of the slow motion, inertia is ineffective.

The general solution to (4.31) is easily found:
$$X_{10}(t_1) = -\left[\frac{C_{\mathrm{g}}}{(gh)^{\frac{1}{2}}}\,R^2+(1-R^2)\right]\frac{C_{\mathrm{g}}}{\rho h(gh-C^2_{\mathrm{g}})}\,\mathrm{e}^{-\beta t_1}\int_0^{t_1} S(-C_{\mathrm{g}}\,\tau)\,\mathrm{e}^{\beta\tau}\,\mathrm{d}\tau, \tag{4.32}$$

where $$\beta \equiv \frac{K}{2\rho h (gh)^{\frac{1}{2}}}. \tag{4.33}$$

To get $\bar{\zeta}_{20}^-$ for any $x_- < 0$, we simply replace t_1 by $t_1 + x_1/(gh)^{\frac{1}{2}}$ in every term of (4.26*a*). Similarly, for all $x_1 > 0$, we replace all t_1 by $t_1 - x_1/(gh)^{\frac{1}{2}}$ in every term of (4.26*b*) to get $\bar{\zeta}_{20}^+$.

We can see immediately that, if the mooring stiffness K increases, it takes less time for the slow motion to reach equilibrium under persistent forcing, or to decay when forcing is removed.

5. Special cases

Let us specify the incident envelope to be such that

$$a_0^-(0, t_1) = A_0 = \begin{cases} \hat{A} \sin \Omega t_1 & (0 < t_1 < T_1), \\ 0 & \text{otherwise}, \end{cases} \tag{5.1}$$

where $\hat{A}$ is the maximum amplitude of the incident-wave potential; then the slow displacement is

$$X_{10}(t_1) = \begin{cases} -\hat{X}_{10}\left\{1 - \mathrm{e}^{-\beta t_1} - \mathrm{Re}\left[\left(1 - \dfrac{2\mathrm{i}\Omega}{\beta}\right)^{-1} (\mathrm{e}^{-2\mathrm{i}\Omega t_1} - \mathrm{e}^{-\beta t_1})\right]\right\} & (0 < t_1 < T_1), \quad (5.2) \\ X_{10}(T_1)\, \mathrm{e}^{-\beta(t_1 - T_1)} \quad (t_1 > T_1), & \quad (5.3) \end{cases}$$

where $$\hat{X}_{10} = \left(\frac{C_g}{(gh)^{\frac{1}{2}}} R^2 + 1 - R^2\right) \frac{(gh)^{\frac{1}{2}} C_g}{K} \frac{\hat{S}}{gh - C_g^2} > 0, \tag{5.4}$$

$$\hat{S} = \tfrac{1}{2}\rho g \hat{\mathscr{A}}^2 \left(\frac{2C_g}{C} - \frac{1}{2}\right) \tag{5.5a}$$

and $$\hat{\mathscr{A}} \equiv \left|\frac{2\mathrm{i}\omega \hat{A}}{g} f_0(0)\right|. \tag{5.5b}$$

Note that $\hat{X}_{10}$ is independent of Ω, is inversely proportional to the spring constant K, and is positive. Figure 3 shows its dependence on $k_0 h$ and on $\mathscr{M}$. Let us consider three subcases.

5.1. *A sinusoidal envelope*

If we let $T_1 \uparrow \infty$ in (5.1) then the incident wave envelope is sinusoidal for all $t_1 < \infty$. The limit for quasi-steady state $\Omega t_1 \gg 1$ is

$$X_{10}(t_1) = -\hat{X}_{10} - \hat{X}_{10}\left[1 + \left(\frac{2\Omega}{\beta}\right)^2\right]^{-\frac{1}{2}} \cos 2\Omega(t_1 - \tau) \quad (\Omega t_1 \to \infty), \tag{5.6a}$$

where τ is the phase angle:

$$\tan \tau \equiv \frac{2\Omega}{\beta}. \tag{5.6b}$$

The ratio β/Ω measures the importance of mooring relative to radiation damping. Thus X_{10} oscillates about the mean $-\hat{X}_{10}$. In figure 3, the dimensionless $\hat{\mathscr{X}}_{10} = \hat{X}_{10} h/\hat{\mathscr{A}}^2$ is plotted for three values of $\mathscr{M}$. For comparison the first-order high-frequency displacement X_{11} is also shown. Relative to the mean, the amplitude of the oscillatory part is

$$\left[1 + \left(\frac{2\Omega}{\beta}\right)^2\right]^{-\frac{1}{2}}, \tag{5.7}$$

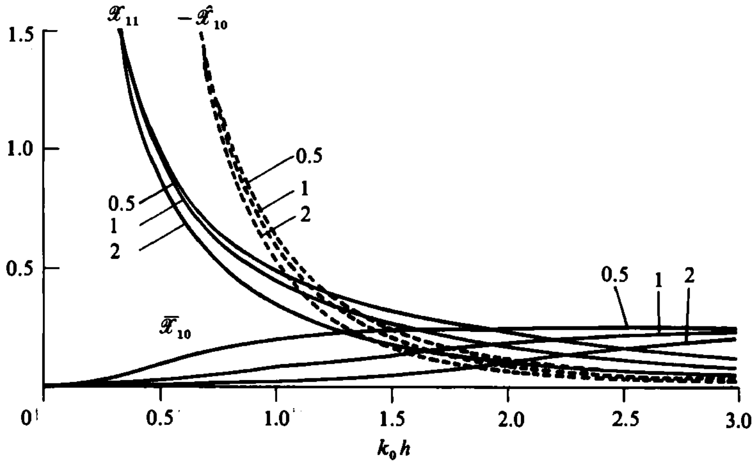

FIGURE 3. Normalized amplitudes of the high frequency displacement $\mathscr{X}_{11} = X_{11}/\mathscr{A}$ and one-half of the normalized drift displacement at steady state for a uniform wavetrain $-\hat{\mathscr{X}}_{10} = \hat{X}_{10} h/\mathscr{A}^2$ (see (5.4)). $\mathscr{M} = 0.5, 1, 2$. The elastic mooring constant is $K = \rho \bar{g} h$. For comparison, $\bar{\mathscr{X}}_{10}$ of (5.11) is also plotted.

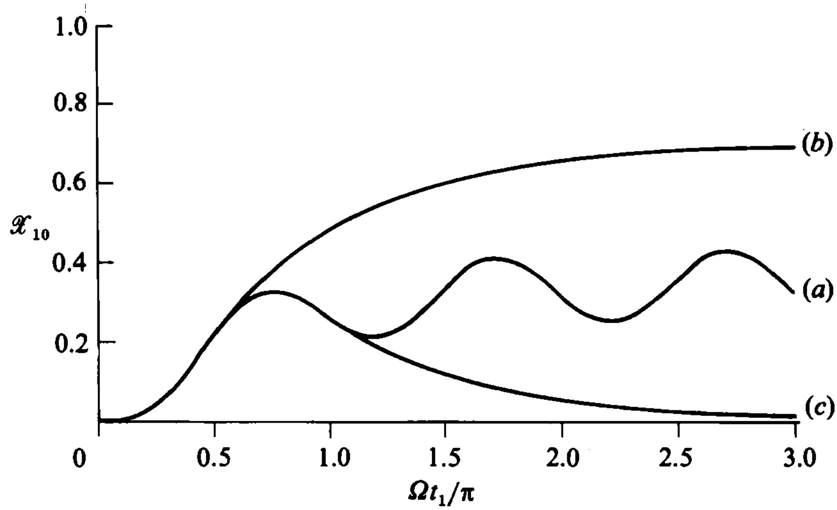

FIGURE 4. Transient slow displacement $-\mathscr{X}_{10}$ of the body due to various types of incident wave envelope: (a) a sinusoidal envelope starting from rest; (b) a uniform envelope starting from rest; (c) a pulse envelope. For all cases, $\Omega = \omega$, $K = \rho g h$, $M = \rho h^2$.

which decreases monotonically from 1 to 0 as β/Ω increases. Thus for tighter mooring the body wanders less. A typical history of $X_{10}(t)$, normalized by $\mathscr{A}^2/h$, is shown by curve (a) in figure 4. Throughout figure 4 we have chosen the modulation frequency to be ϵ times the short-wave frequency, namely $\Omega = \omega$. We have also set $\mathscr{M} = 1$ and $\bar{K} = \epsilon \rho g h$.

The reflected ζ_{20}^{R} and transmitted ζ_{20}^{T} travel outwards at the group velocity. Their amplitudes depend only on R, and are plotted in figure 5(a) for three different values of $\mathscr{M}$. The radiated long waves, which travel at the speed $(gh)^{\frac{1}{2}}$, depend further on K through X_{10t_1}; their amplitudes are equal by virtue of (4.27), and are plotted in figure 5(b) only for $K = \rho g h$.

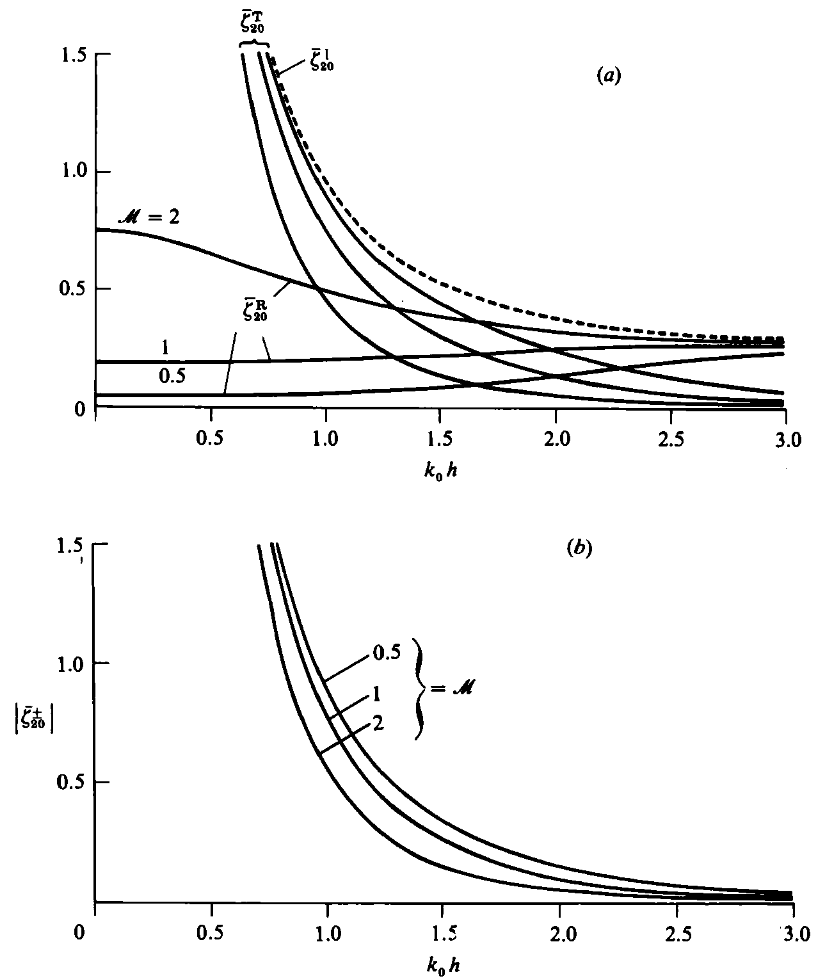

FIGURE 5. (*a*) Normalized amplitudes of the long waves accompanying the incident group $\bar{\zeta}^{I}_{20}$, the reflected group $\bar{\zeta}^{R}_{20}$, and the transmitted group $\bar{\zeta}^{T}_{20}$. (*b*) Normalized amplitude of the radiated long waves travelling at the speed $(gh)^{\frac{1}{2}}$ to the right $\bar{\zeta}^{+}_{20}$ and to the left $\bar{\zeta}^{-}_{20}$. Normalization length is $\mathscr{A}^2/h$.

5.2. *A constant envelope*

If we let the incoming wave have the form

$$A_0 = \begin{cases} \hat{A} \sin \Omega t_1 & (0 < t_1 < \pi/2\Omega), \\ \hat{A} & (\pi/2\Omega < t_1), \end{cases} \tag{5.8}$$

we get in (4.31) a constant forcing term when $t_1 > \pi/2\Omega$, and the solution tends asymptotically to $X_{10} = -2\hat{X}_{10}$ as follows:

$$X_{10} = -2\hat{X}_{10} + \left[2\hat{X}_{10} + X_{10}\left(\frac{\pi}{2\Omega}\right)\right] \mathrm{e}^{-\beta(t_1 - \pi/2\Omega)}. \tag{5.9}$$

For $0 < t_1 < \pi/2\Omega$ the solution $X_{10}(t_1)$ is still given by (5.2). Note that X_{10} is negative, implying that the drift displacement is *opposite* to the direction of the incident waves.

When a cylinder is floating or immersed so that fluid is allowed to pass from one side to the other, the drift force due to normally and steadily incident waves is known to be

$$\rho g \hat{\mathscr{A}}^2 R^2 C_g / C \tag{5.10}$$

and is in the *same* direction as the incident waves (see e.g. Longuet-Higgins 1977). The corresponding displacement is just

$$\bar{X}_{10} = \rho g \hat{\mathscr{A}}^2 R^2 C_g / CK. \tag{5.11}$$

It is different both in sign and in magnitude (see figure 3) from $-2\hat{X}_{10}$, the steady-state limit of X_{10} (see (5.9)). Does this contradict our result?

Observe first that (5.11) can be got from (4.28) if one sets

$$\Delta\phi_{10t_1} = 0. \tag{5.12}$$

In the long-time limit of a uniform wavetrain, the long wave described by ϕ_{10} becomes a steady current. For a two-dimensional body that does not prohibit steady flow of water from one side to the other, (5.12) holds when viscous effects are ignored. Now, in the present example, $\Delta\phi_{10t_1}$ does not vanish; indeed, it also contributes a new part in the jump of mean sea level across the body, since

$$g\,\Delta\bar{\zeta}_{20} = -\Delta\phi_{10t_1} - \Delta[\overline{|\phi_{11x}|^2} + \overline{|\phi_{11z}|^2} - \overline{\sigma(\phi_{11}\phi^*_{11z} + *)}]. \tag{5.13}$$

The second part $\Delta[\ ]$ above can be calculated

$$\frac{1}{2}\frac{g\hat{\mathscr{A}}^2}{h} R^2 \left(\frac{2C_g}{C} - \frac{1}{2}\right), \tag{5.14}$$

which is precisely the mean-sea-level change across a two-dimensional body if fluid passage is allowed (Longuet-Higgins 1977).

To confirm our result, let us give a more physical derivation of the steady drift force on a fixed plate that seals off all communications between two fluid regions $x > 0$ and $x < 0$. We assume that the envelope of the incident wavetrain grows steadily from zero to a constant value. The fluid on the right is never disturbed. In the fluid on the left the steady-state spatial average of the normal radiation stress in the standing wave is known to be

$$\bar{S}_{xx} = 2\hat{E}\left(\frac{2C_g}{C} - \frac{1}{2}\right) = 2\hat{S}, \quad \text{where } \hat{E} = \tfrac{1}{2}\rho g \hat{\mathscr{A}}^2. \tag{5.15}$$

The steady-state mean setdown is also twice that of the incident progressive wave, so that the corresponding hydrostatic pressure is

$$\rho g \bar{\zeta}_{20} = -\frac{2\hat{S}}{1 - C_g^2/gh}. \tag{5.16}$$

The sum of the two gives the drift force on the plate as $t_1 \to \infty$:

$$\rho g \bar{\zeta}_{20} + \bar{S}_{xx} = -2\hat{S}\left(\frac{1}{1 - C_g^2/gh} - 1\right) = -2\hat{S}\left(\frac{C_g^2}{gh - C_g^2}\right). \tag{5.17}$$

On the other hand, if we take $t_1 \to \infty$ in (5.9) and $R = 1$ in (5.4), we get the steady-state displacement whose product with K is precisely equal to the steady-drift force given by (5.17). Thus the mean setdown on the upwave side is responsible for the negative drift force.

In figure 4 the typical transient motion $X_{10}(t_1)$ is plotted as curve (*b*) for the case where the incident envelope becomes uniform after the first peak at $\Omega t_1 = \frac{1}{2}\pi$.

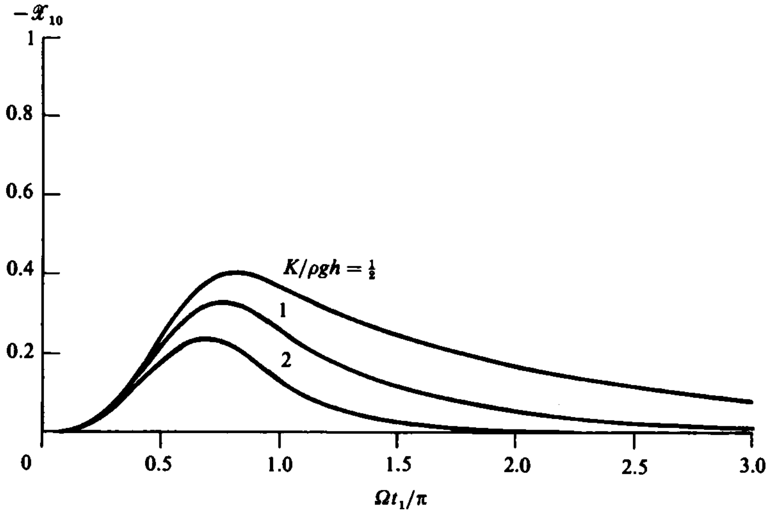

FIGURE 6. Effects of elastic mooring constant on the transient slow displacement $-\mathscr{X}_{10}$ of the body due to a wave packet for $M = \rho h^2$ and $k_0 h = 1.25$.

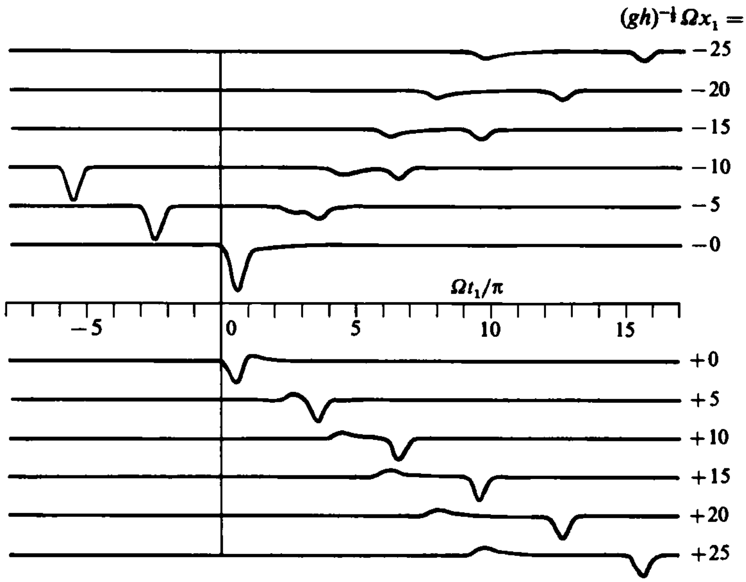

FIGURE 7. Scattering and radiation of long waves due to an incident wave packet; $k_0 h = 1.25$, $\mathscr{M} = 1$, $K = \rho g h$.

5.3. *A pulse envelope*

Let the pulse envelope have the total duration $T_1 = \pi/\Omega$. After the pulse expires ($\Omega t_1 > \pi$), slow sway of the body gradually attenuates, as shown by curve (*c*) in figure 4. The rate of attenuation increases with K through β; see (5.3). The maximum of $X_{10}(t_1)$ lags behind the peak of the incident-wave envelope at $\Omega t_1 = \frac{1}{2}\pi$. This is due to the time constant $1/\beta \propto 1/K$.

To see the effect of the mooring force, we plot in figure 6 the effect of β/Ω for the same wave packet. For smaller β/Ω (or K), the maximum displacement is greater but is realized later.

The long waves are particularly interesting for this case. As shown in figure 7, ζ_{20}^{I}, which accompanies the incident-wave packet, is a solitary depression (setdown) travelling at the group velocity. The reflected and transmitted wave packets are also accompanied by setdowns $\bar{\zeta}_{20}^{R}$ and $\bar{\zeta}_{20}^{T}$, whose amplitudes are simply related to the reflection coefficient R. Ahead of them are the two long waves $\bar{\zeta}_{20}^{\pm}$ travelling at speed $(gh)^{\frac{1}{2}}$. Because $X_{10}(t)$ is pulselike, X_{10t_1} and $\bar{\zeta}_{20}^{\pm}$ must have points of inflection (cf. (4.26)).

6. Concluding remarks

By a multiple-scale analysis, we have been able to separate the low-frequency part of the second-order motion of a moving block from the high-frequency part. Because of the analytical simplicity, the physics of the body motion and the associated wave dynamics are now more easily understood. For two- or three-dimensional floating bodies of arbitrary shapes and a limited draught, our approach should offer similar advantages, although some numerical computation is needed. For ships and tension-leg platforms, the case of large drift displacement (drift velocity comparable to wave orbital velocity) is of interest; both full physical understanding and an effective method of analysis are still lacking.

Preliminary results of this paper were presented at the International Workshop on Ship and Platform Motions at Berkeley on 26–28 October 1983. This research has been sponsored by the Office of Naval Research (Contract N00014-BO-C-0531) and the National Science Foundation (Grant MEA 77-17817-A04).

REFERENCES

FALTINSEN, O. M. & LØKEN, A. E. 1979 Slow drift oscillations of a ship in irregular waves. *Appl. Ocean Res.* 1, 21–31.

HSU, F. H. & BLENKARN, K. A. 1970 Analysis of peak mooring forces caused by slow vessel drift oscillations in random seas. *Proc. Offshore Technol. Conf., Houston; Paper* 1159.

LONGUET-HIGGINS, M. S. 1977 The mean forces exerted by waves on floating or submerged bodies, with applications to sand bars and wave-power machines. *Proc. R. Soc. Lond.* A **352**, 463–480.

LONGUET-HIGGINS, M. S. & STEWART, R. H. 1964 Radiation stress in water waves; a physical discussion with application. *Deep-Sea Res.* **11**, 529–562.

MEI, C. C. 1983 *The Applied Dynamics of Ocean Surface Waves.* Wiley-Interscience.

MOLIN, B. & BUREAU, G. 1980 A simulation model for the dynamic behavior of tankers moored to single point moorings. In *Proc. Intl Symp. on Ocean Engineering Ship Handling, Swedish Maritime Research Center, Gothenburg.*

NEWMAN, J. N. 1974 Second order, slowly varying forces on vessels in irregular waves. In *Proc. Intl Symp. on Dynamics of Marine Vehicles and Offshore Structures in Waves, University College London.*

OGILVIE, T. F. 1983 Second-order hydrodynamic effects on ocean platforms. *Paper presented at Intl Workshop on Ship and Platform Motions, University of California at Berkeley.*

PINKSTER, J. A. 1976 Low frequency second order wave forces on vessels moored at sea. In *Proc. 11th Symp. on Naval Hydrodynamics, University College London.*

PINKSTER, J. A. 1980 Low frequency second order wave exciting forces on floating structures. *Netherlands Ship Model Basin Publ.* 650.

REMERY, G. F. M. & HERMANS, A. J. 1971 The slow drift oscillations of a moored object in random seas. In *Proc. Offshore Technol. Conf., Houston.*

TRIANTAFYLLOU, M. S. 1982 A consistent hydrodynamic theory for moored and positioned vessels. *J. Ship Res.* **26**, 97–105.